

Imperatives and Logic*

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Abstract

In this paper we present a series of dynamic semantics for imperatives in the framework of update semantics, and specify several alternative logics for imperatives. In these semantics the consistency problem and Ross's paradox have straightforward solutions. All the work is carried out based on the correspondence between imperatives and force structures. According to different ways of dealing with compatibility between imperatives, we give a few choices of defining the notion of consistency for imperatives. Meaning of imperatives is an update function on force structures, which is also dependent on compatibility. Entailment for imperatives is reduced to some relation between force structures. We also give some choices of the notion of entailment, each of which is coincident with a certain kind of compatibility

1 Introduction

1.1 Imperatives

In English imperatives are roughly characterized as sentences which do not have overt subjects, and have verbs in the bare form¹. Imperatives may express different speech acts, such as command, prohibition, request, and even curse. Here are some examples²:

- (1) a. Pull over your car! (*command*)
- b. Don't go to the party! (*prohibition*)
- c. Help me with my bicycle! (*request*)
- d. Have fun! (*wish*)
- e. Go to hell! (*curse*)

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¹See Han [3] for extensive refinements and arguments about this characterization. This characterization may have some problems, and see Russell [9] for more discussions.

²Some of these examples are taken from Schwager [10]. Note that which speech act an imperative expresses is context-dependent.

But generally speaking, imperatives are used by speakers to change the world by influencing the behaviors of addressees, not to describe the world. In this sense, they don't have truth values. This is one of the main differences between indicatives and imperatives.

Imperatives have propositional contents, which can be expressed by indicatives. It is difficult to give a specific order (or request, etc.) without any propositional content³. The following indicatives can respectively express the propositional contents of the imperatives in (1):

- (2) a. You will pull over your car.
- b. You will not go to the party.
- c. You will help me with my bicycle.
- d. You will have fun.
- e. You will go to hell.

We think that each imperative contains two factors: imperative force and propositional content. They play different roles for imperatives: The first factor indicates that something is commanded (or requested, etc.), and the second factor indicates what is commanded (or requested, etc.). Similar arguments can be found in Han [3] and Jorgensen [5]. In this paper, we use $I(\phi)$ to represent an imperative, where ϕ expresses a proposition.

In natural language, any boolean combination of indicatives is still an indicative. In other words, the set of indicatives is closed under the boolean connectives. But for imperatives, this is not the case. Actually the set of imperatives is not closed under any connective except conjunction.

- (3) a. # Not close the door!
- b. # Do close the door! or do close the window!
- c. # If close the door! then close the window!
- d. Close the door! and close the window!

Imperatives can not be negated. At least in English, any sentence of the form $\neg I(\phi)$ is infelicitous, as shown by (3a). Furthermore, neither disjunction nor implication can take two imperatives as its arguments. However, conjunctions of imperatives, such as (3d), are often meaningful. In this paper we view a conjunction of imperatives as a sequence of imperatives.

However, not all boolean combinations containing imperatives are infelicitous: Some of them in mixed moods may be meaningful. Here are some examples:

- (4) a. Leave your phone number and I will call you tomorrow.
- b. Show your pass or I will not let you in.
- c. If you see John some day, tell him this news.

Since we only focus on imperatives in this paper, we put aside these cases and leave developing a semantics for them to another occasion.

³Actually in some special cases, people do this. For example, a mother may make such a convention with her son: A cough from her represents the order: Go to bed now! But we simply ignore these cases in this paper.

1.2 Problems

This paper aims to solve two problems: consistency of imperatives and Ross’s paradox. We focus on consistency problem firstly. Look at the following three sequences of imperatives:

- (5) a. Close the door or the window!
b. Don’t close the door!
- (6) a. Close the door or the window!
b. Don’t close them both!
- (7) a. Close the door or the window!
b. Close the door!

Suppose that in any sequence, the imperatives are uttered by different speakers. If we only consider the propositional contents of these imperatives, all of these sequences are classically consistent, simply because they each have a classical model. But intuitively it seems that any of them more or less contains some conflict. How should we account for the intuitions about this?

We move to the second problem. The following inference is called Ross’s paradox, which was mentioned in Ross [8]:

- (8) Slip the letter into the letter-box! \models Slip the letter into the letter-box or burn it!

This inference is valid in classic logic, but weird in our intuitions. We think that a successful semantics for imperatives should avoid this inference to be valid. In what follows we will present a semantics for imperatives, in which these problems have straightforward solutions.

In the literature there are two main directions in which the semantics for imperatives was developed. The work in the first direction considers imperatives having truth values, and uses STIT logic to deal with them. The related work includes Horty [4], Segerberg [11], Aloni [1], Schwager [10], etc.. The second direction was inspired by linguistic semantics, and the related work includes Portner [7], Mastop [6], Veltman [13], etc..

Our work is along the second direction. We present the semantics in the framework of update semantics, which was proposed by Veltman [12]. In this semantics, meaning of imperatives is an update function on force structures, which essentially share the similar idea with plans defined in Veltman [13].

After briefly stating the basic theory of update semantics in section 2, we introduce force structures, for which we also define paths, routes and tracks. We analyze the consistency problem in detail in section 3. The semantics is depicted in section 4. In section 5, we define the validity of inferences for imperatives. Section 6 is the conclusion of this paper. We put the proofs of some propositions occurring in this paper to the appendix section.

2 Force structures

Firstly we state the basic idea of update framework. An update system is a triple $\langle \mathcal{L}, \Sigma, [\cdot] \rangle$, where \mathcal{L} is the language for which we define the semantics, Σ is the set

of information states, and $[\cdot]$ is a function from \mathcal{L} to $\Sigma \rightarrow \Sigma$, which assigns to each sentence ϕ an operation $[\phi]$, which is from information states to information states. For any ϕ , $[\phi]$ is called an update function. The elements of Σ can be viewed as contexts. In this sense, meaning of language lies in how it updates the contexts. In the update framework, the validity is defined based on the acceptance relation \vdash , which is between information states and sentences. For any information state σ and any sentence ϕ , $\sigma \vdash \phi$ if the information conveyed by ϕ is already subsumed by σ .

We turn to imperatives. As we mentioned previously, any imperative contains two factors: propositional content and imperative force. Intuitively, uttering an imperative gives some imperative force to the agent, which tends to push him to make true the propositional content of this imperative in the future. This suggests that we can deal with imperatives in the update framework in this way: Identify information states with states of imperative forces born by the agent; The meaning of imperatives lies in how they change the force states; A force state accepts an imperative if the force brought about by this imperative is already contained in this force state. This is the basic idea of the semantics presented in this paper. Now we start to formalize this idea.

2.1 Force structures

Let \mathcal{Y} be the standard language for the classical propositional logic, which is defined as follows.

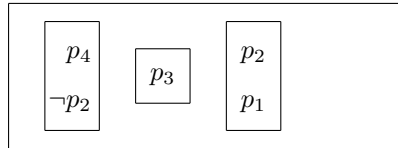
Definition 1. (Language \mathcal{Y})

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi$$

Let $\mathcal{L} = \{I(\phi) \mid \phi \in \mathcal{Y}\}$ be the set of all imperatives. \mathcal{L} is the language for which we will give a semantics.

Let A be the set of literals of \mathcal{Y} . We call a literal l as a force. Let $B = \{X \subseteq A \mid X \text{ is finite}\}$. Each $J \in B$ is called a choice scope. Note that the empty set \emptyset is also a choice scope. Let $F = \{X \subseteq B \mid X \text{ is finite}\}$. Each $K \in F$ is a force structure. The empty set \emptyset is called the minimal force structure. Those force structures containing \emptyset are called as absurd ones.

Each force structure describes a state of imperative forces exerted to the agent. We explain the reading of force structures by an example. Look at the force structure $K_1 = \{\{p_4, \neg p_2\}, \{p_3\}, \{p_2, p_1\}\}$, as illustrated in the following picture⁴:



In this picture, there are three small blocks in a big block. Each small block is a choice scope. For each choice scope, the agent has to choose at least one literal in it and make this literal true. The agent has no choices for single choice scopes, and must make true the only literal in them. For non-single choice scopes, he has free choices. Note this freedom is relative. For example, if the agent chooses $\neg p_2$ in the first choice scope, then he must choose p_1 in the last one, since in this case he can not make true p_2 any

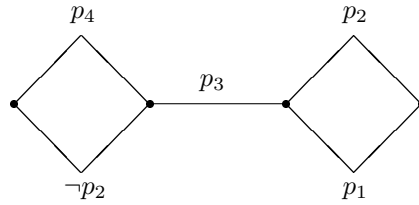
⁴Note that we didn't put any order between the choice scopes for any force structure.

more. It should be noted that not all literals in A occur in this force structure, which only includes those literals having imperative forces to the agent.

For any choice scope, making true more than one literal in it is allowed absolutely. For any choice scope, if one literal of it is made true by the agent, we say that this choice scope is performed. If all choice scopes in a force structure are performed, we say that this force structure is performed.

2.2 Paths, routes and tracks

Let $K = \{X_1, \dots, X_n\}$ be any force structure. Define the set C as follows: $C = \{\langle l_1, \dots, l_n \rangle \mid \langle l_1, \dots, l_n \rangle \in X_1 \times \dots \times X_n\}$. Each $P \in C$ is called a path of K . P is consistent if and only there is no propositional variable p such that, both p and $\neg p$ are in P . Each path is a way to perform K . The following picture shows the paths of K_1 , as mentioned above.

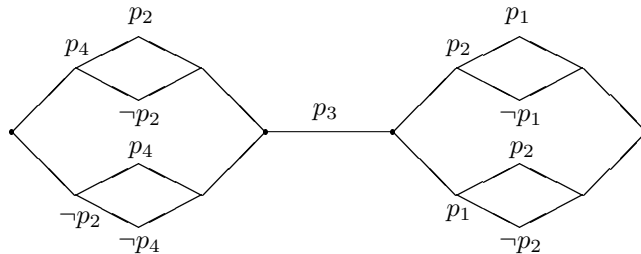


Every road from the leftmost point to the rightmost point is a path of K_1 ⁵. We can see that p_3 appears in every road, which means that the agent must make it true.

For any $X_i \in K$, the addressee may choose more than one literal in it. Thus the union of some paths is also a way of performing K . Generally we define the set of all ways of performing K . Let $D = \{\bigcup Z \mid Z \subseteq C\}$. Each $R \in D$ is called a route of K . We can see that each path is a route. R is consistent if and only if it does not contain any contradiction.

Now we define tracks for K , whose definition is a bit complicated. For any X_i , let X'_i be the smallest set such that both p and $\neg p$ are in X'_i for any p occurring in X_i . $T = X''_1 \cup \dots \cup X''_n$ is a track for K if and only if: (1) $X''_i \subseteq X'_i$ and $X''_i \cap X_i \neq \emptyset$; (2) For any p occurring in X_i , one and only one of p and $\neg p$ is in X''_i . T is consistent if and only if it does not contain any contradiction.

The idea for tracks is simple. Each track T completely describes a way to perform K : For any $X_i \in K$, T specifies clearly which literals the agent chooses, and which ones he does not. The set of tracks for K describes all complete ways of performing K . The following picture shows the set of tracks for K_1 .



⁵Note that there is no order.

Every road from the leftmost point to the rightmost one is a track for K_1 .

Routes are different from tracks in two aspects: (1) Routes may not specify which literals the agent does not choose; (2) Routes may not specify which ones he chooses. The first aspect does not need more words. Here is an example for the second aspect: $K_2 = \{\{p_1, \neg p_1, p_2\}\}$. We can see that $\{p_2\}$ is a route for K_2 . The agent has to make a choice from p_1 and $\neg p_1$ ⁶, but the route $\{p_2\}$ does not specify which one. Since the variable p_1 does not appear in $\{p_2\}$, it is not a track. Each track uniquely determines a route, but not vice versa.

For a force structure K , any path is a way to perform this force structure. The agent can freely choose any path to perform K . In this sense we can say that paths of K are free choices to perform K . Each route of K is also a way to perform K . The difference of routes from paths is that paths are “simplest” routes. To perform K , the agent also can freely choose the route he prefers. Tracks of K are different from paths and routes in that they completely describe the ways to perform K . Therefore, we can say that paths, routes and tracks are three different types of free choices to perform force structures. The difference among them is crucial. Later we will see that they produce different definitions of consistency, and consequently produce different notions of entailment.

2.3 Correspondence between imperatives and force structures

Let $I(\phi)$ be any imperative. The indicative ϕ consists of some propositional variables and connectives. The set of these variables generates a set of literals. Each literal can be viewed as an atomic event. In order to perform $I(\phi)$, what the agent needs to do is to make true some literals in this set. There may be some literals which the addressee must make true absolutely, and there may also be some literals which he may not make true conditionally. In what follows we define two functions T^+ and T^- , under which every imperative $I(\phi)$ corresponds to a force structure, which exactly represents the situation that the agent faces in order to perform $I(\phi)$.

F is the set of force structures. Let $K \in F$ be an arbitrary force structure. T^+ and T^- are two functions from $F \times \mathcal{S}$ to F , which are defined by parallel recursion in the following way:

- (a) $T^+(K, p) = \begin{cases} \{\{p\}\} & \text{if } K = \emptyset \\ \{X \cup \{p\} \mid X \in K\} & \text{otherwise} \end{cases}$
 $T^-(K, p) = \begin{cases} \{\{\neg p\}\} & \text{if } K = \emptyset \\ \{X \cup \{\neg p\} \mid X \in K\} & \text{otherwise} \end{cases}$
- (b) $T^+(K, \neg\phi) = T^-(K, \phi)$
 $T^-(K, \neg\phi) = T^+(K, \phi)$
- (c) $T^+(K, \phi \wedge \psi) = T^+(K, \phi) \cup T^+(K, \psi)$
 $T^-(K, \phi \wedge \psi) = T^-(T^-(K, \phi), \psi)$
- (d) $T^+(K, \phi \vee \psi) = T^+(T^+(K, \phi), \psi)$
 $T^-(K, \phi \vee \psi) = T^-(K, \phi) \cup T^-(K, \psi)$

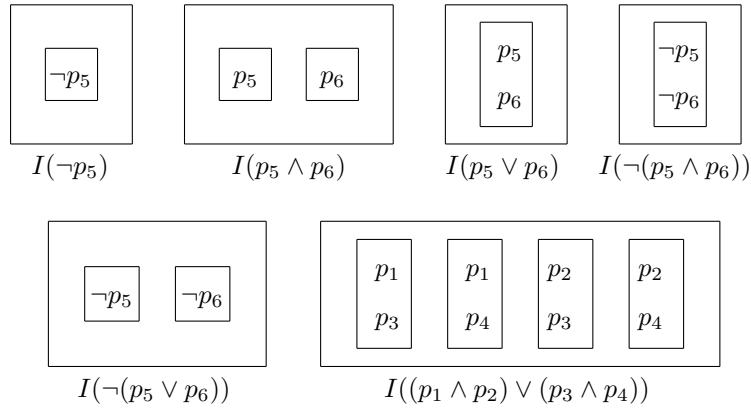
⁶ For any literal, the agent has to make a choice about whether he will make true it. There is no third option. Even if he decides to do nothing, he still makes his choice for this literal.

For any imperative $I(\phi)$, $T^+(\emptyset, \phi)$ is its corresponding force structure.

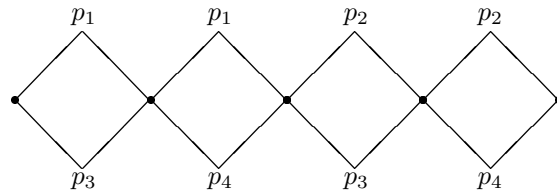
To show why imperatives should correspond to force structures in this way, we give some examples:

- (9) a. Don't open the door! $[I(\neg p_5)]$
 b. Open the door and the window! $[I(p_5 \wedge p_6)]$
 c. Open the door or the window! $[I(p_5 \vee p_6)]$
 d. Don't open the door and the window! $[I(\neg(p_5 \wedge p_6))]$
 e. Don't open the door or the window! $[I(\neg(p_5 \vee p_6))]$
 f. Open switch 1 and 2 or open switch 3 and 4! $[I((p_1 \wedge p_2) \vee (p_3 \wedge p_4))]$

The following force structures correspond to these imperatives respectively:



The imperative (9f) is a bit complicated. To comply with it, the agent has to either open switch 1 and 2 or open switch 3 and 4. The following picture shows all paths of the force structure of (9f):



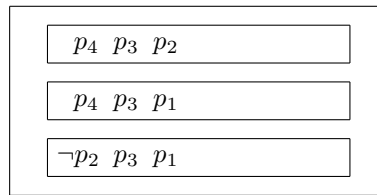
We can see that for each path, either both p_1 and p_2 appear on it or both p_3 and p_4 appear on it.

For any set $\Gamma = \{I(\phi_1), \dots, I(\phi_n)\}$, we say that $T^+(\emptyset, \phi_1) \cup \dots \cup T^+(\emptyset, \phi_n)$ is the force structure corresponding to it. Particularly, the empty set corresponds to itself. We also call the force structure of the set $\{I(\phi_1), \dots, I(\phi_n)\}$ as the force structure of any sequence generated from this set.

Every imperative uniquely corresponds to a force structure under the function T^+ . Such a correspondence is the basis for all of our following work in this paper: Consistency, semantics and logic for imperatives will be given based on this correspondence.

In the introduction section, we said that force structures and plans defined by Veltman [13] share the similar idea. Here we briefly restate his theory. Let A be the set of literals of \mathcal{L} . Each literal can be viewed as an atomic event. Let $B = \{X \subseteq A \mid X \text{ is finite}\}$. Each $J \in B$ is called a to-do list. A to-do list J is consistent if and only if there is no variable $p \in \mathcal{L}$ such that both p and $\neg p$ are in J . A finite set of consistent to-do lists is called a plan.

Essentially there is no difference between the definitions of force structures and plans, except that plans only consists of consistent sets of literals. However, plans have a completely different reading from force structures. We show it by an example. Look at the following picture, which represents the plan $\Pi_1 = \{\{p_4, p_3, p_2\}, \{p_4, p_3, p_1\}, \{\neg p_2, p_3, p_1\}\}$:



In this plan, each block is a to-do list. The agent has to choose at least one to-do list in order to complete this plan, but he can freely choose any one of them. For the to-do list chosen, he has to make true all literals in it. The literal p_3 occurs in every to-do list, which means the agent must make it true. All literals except p_3 do not occur in all to-do lists, which means that he gets explicit permission to make them true, but he may not to do that. Clearly not all literals in A occur in this plan, and it only includes those on which the agent has got explicit constraints. All literals not in this plan are the events not mentioned by the speaker.

In Veltman's theory, meaning of imperatives lies in changing plans in some way. We will come back to this theory later, and contrast it with ours.

3 Consistency

Let $\langle I(\phi_1), \dots, I(\phi_n) \rangle$ be any sequence of imperatives. Suppose that these imperatives are uttered one by one by some speakers to the agent. In this section we focus on the question: Are these utterances consistent?

Let $\langle \psi_1, \dots, \psi_m \rangle$ be any sequence of indicatives. In the classical semantics of indicatives, $\langle \psi_1, \dots, \psi_m \rangle$ is consistent if and only if the set $\{\psi_1, \dots, \psi_m\}$ has a classical model. This definition implies that $\langle \psi_1, \dots, \psi_m \rangle$ is consistent if and only if $\psi_1 \wedge \dots \wedge \psi_m$ is consistent, which means that consistency of sequences can be reduced to consistency of single indicatives. However, whether we should define consistency for imperatives in a similar way is not easy to answer, even if we only consider their propositional contents.

3.1 Single imperatives

Firstly we think that this definition is also applicable to single imperatives. Let $I(\phi)$ be any imperative. The consistency of $I(\phi)$ is defined as follows.

Definition 2. (Consistency of imperatives) The imperative $I(\phi)$ is consistent if and only if ϕ has a classical model.

We see that $I(\phi)$ is consistent if and only if its force structure contains a consistent path. As the proposition content of $I(\phi)$, ϕ describes a fact about the world, which may have not been realized yet but is wished to be true by the speaker. When the agent makes ϕ true in the future, this imperative is performed. There is no constraints on what is the world described by ϕ like, unless it is an impossible world.

There are some imperatives, which satisfy this condition but seem weird. The imperative (10) is such an example. However, we think that this sort of weirdness comes from the perspective of pragmatics, just like (11) is weird, but still classically consistent.

(10) Kick the red ball or the blue ball and don't kick the red ball!

(11) It is raining or snowing and it is not raining.

In order to keep coincident with this argument, we define the consistency of force structures as follows, where we count the empty set as a consistent force structure:

Definition 3. (Consistency of force structures) The force structure K is consistent if and only if $K = \emptyset$ or it has a consistent path.

In this way $I(\phi)$ is consistent if and only if its force structure is consistent. The following definition is too strong: The force structure K is consistent if and only if any path of it is consistent. According to it, many ordinary imperatives in natural language are not consistent. The imperative (12) is such an example.

(12) Close the door or the window but don't close them both!

Therefore we consider this definition unreasonable.

3.2 Restrictions between imperatives

The situation gets more complicated when we generally consider sequences of imperatives. Here the problem is: Should we require that the sequence $\langle I(\phi_1), \dots, I(\phi_n) \rangle$ is consistent if and only if the imperative $I(\phi_1 \wedge \dots \wedge \phi_n)$ is consistent? In deed, this problem involves where we should draw the borderline between semantics and pragmatics for imperatives.

We generally talk about the restrictions between utterances. We focus on uttering indicatives firstly. We say that two utterances are restricted by each other if (1) There is a classical model in which both of them are true; (2) They both satisfy the maxim of quantity in the sense of Grice [2]; (3) From the two utterances we can get that the two speakers of them are in different information states. Here is an example. Suppose (13a) and (13b) are uttered by the speakers A and B respectively:

- (13) a. It is raining or snowing.
b. It is not raining.

Suppose both of the utterances satisfy the maxim of quantity. From (13a) we get that, the speaker A does not know whether it is raining, and does not know whether it is snowing either, although he knows that at least one of them is true. However, the

speaker B of (13b) knows whether it is raining. We see that the two speakers have different information states about the world, and there is some conflict between the two utterances. But (13a) and (13b) may be true at the same time. In this sense we say that they are restricted by each other.

There is also a similar problem with imperatives. We consider utterance of imperatives. We say that two utterances are restricted by each other if (1) The propositional contents of the two imperatives may be true in a model; (2) By each imperative, what the speaker exactly wishes is that the propositional content of it would be true; (3) From the two utterances, we can get that the two speakers of them have different mind states. We give an example. Suppose (14a) and (14b) are uttered by the speakers A and B respectively:

- (14) a. Close the door or the window!
 b. Don't close the door!

From (14a), we know what the speaker A really wishes is that the door or window would be closed by the addressee, and he does not care which one. Therefore closing the door is fine with the speaker A. But closing the door is not fine with the speaker B according to (14b). This implies that the speakers A and B have different mind states.

The classical definition of consistency for indicatives does not consider this sort of restrictions, and just delegate them to pragmatics. However intuitively counting sequences like (14) as consistent ones is more strange than counting sequences like (13) as consistent. It makes more sense to put these restrictions between utterances of imperatives into semantics. According to where we draw the borderline between semantics and pragmatics, there are two different directions in which we define consistency for imperatives: (1) The restrictions between utterances are allowed, and $\langle I(\phi_1), \dots, I(\phi_n) \rangle$ is consistent if and only if $I(\phi_1 \wedge \dots \wedge \phi_n)$ is consistent; (2) Consistent utterances don't contain any restrictions. In the coming, we will define consistency along each of the two directions. In particular, we will give some alternative choices in the second direction.

3.3 Sequences of imperatives

In the first direction, the definition of consistency of sequences is straightforward, because we have defined consistent imperatives.

Definition 4. (Consistency₁ of sequences) The sequence $\langle I(\phi_1), \dots, I(\phi_n) \rangle$ is consistent if and only if the force structure of $\{I(\phi_1), \dots, I(\phi_n)\}$ has a consistent path.

According to this definition, only sequences like (15), which there is no way to perform, are inconsistent. Sequences containing restrictions like (14) are consistent.

- (15) a. Drink milk!
 b. Don't drink milk!

We turn to the second direction, in which the consistency is sensitive to the restrictions between utterances. In what follows, we use compatibility between force structures to capture the restrictions between imperatives, in the sense that two consistent imperatives are restricted by each other if and only if their force structures are not compatible. Actually we don't have clear intuitions about consistency of imperatives. Considering this point, we give three alternative choices of compatibility, among which

the extent of allowed restrictions are different. Later we will see that these different notions of compatibility will produce different notions of consistency. Let K_1 and K_2 be any force structures.

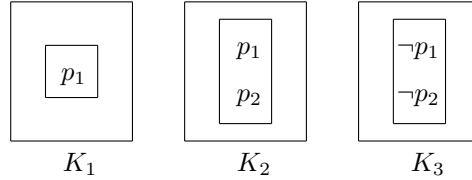
Definition 5. (Compatibility₂ of force structures) The force structures K_1 and K_2 are compatible₂ if and only if (1) For any consistent path P_1 of K_1 , there is a consistent path P of $K_1 \cup K_2$ such that $P_1 \subseteq P$; (2) For any consistent path P_2 of K_2 , there is a consistent path P of $K_1 \cup K_2$ such that $P_2 \subseteq P$.

Definition 6. (Compatibility₃ of force structures) The force structures K_1 and K_2 are compatible₃ if and only if (1) For any consistent route R_1 of K_1 , there is a consistent route R of $K_1 \cup K_2$ such that $R_1 \subseteq R$; (2) For any consistent route R_2 of K_2 , there is a consistent route R of $K_1 \cup K_2$ such that $R_2 \subseteq R$.

Definition 7. (Compatibility₄ of force structures) The force structures K_1 and K_2 are compatible₄ if and only if (1) For any consistent track T_1 of K_1 , there is a consistent track T of $K_1 \cup K_2$ such that $T_1 \subseteq T$; (2) For any consistent track T_2 of K_2 , there is a consistent track T of $K_1 \cup K_2$ such that $T_2 \subseteq T$.

The three definitions share the same idea: Two force structures are compatible if and only if no consistent path (route or track) of them will be lost after putting the two force structures together. Paths, routes and tracks are three types of free choices of force structures. Hence what the three definitions say is that there is no conflicts among the free choices of two compatible force structures. By doing so, we catch the restrictions between imperatives. In what follows we also say that two imperatives are compatible if their force structures are compatible.

As we have mentioned, every consistent path is a consistent route, and any consistent route can be extended to a consistent track. Therefore we have the following results about the strength of the three compatibilities: $compatibility_2 \leq compatibility_3 \leq compatibility_4$. Actually this order is strict, which we show by some examples:



It is easy to verify that K_1 and K_2 are compatible₃ but not compatible₄. Hence, we have that $compatibility_3 < compatibility_4$. We also get that K_2 and K_3 are compatible₂ but not compatible₃. Therefore $compatibility_2 < compatibility_3$.

We point out some special facts implied by these notions of compatibility. The first one is: Any two inconsistent force structures are compatible under any of these definitions, simply because neither of them contains any consistent paths, routes or tracks. Hence the conditions of compatibility are trivially satisfied. The second fact is: As a consistent force structure, the empty set \emptyset is compatible with any force structure relative to any of these notions of compatibility. The third one is: The absurd force structure $\{\emptyset\}$ is only compatible with inconsistent force structures and \emptyset , since $K \cup \{\emptyset\}$ does not contain any consistent paths, routes or tracks for any K .

We define the consistency of the sequence $\langle I(\phi_1), \dots, I(\phi_n) \rangle$ by following such an intuition: A sequence of utterances is consistent if and only if each utterance in it is

compatible with all previous utterances as a whole. The three notions of compatibility generate three notions of consistency:

Definition 8. (Consistency₂ of sequences of imperatives) The sequence $\langle I(\phi_1), \dots, I(\phi_n) \rangle$ is consistent₂ if and only if (1) The force structure of the set $\{I(\phi_1), \dots, I(\phi_n)\}$ has a consistent path; (2) The force structure of ϕ_i is compatible₂ with the force structure of the set $\{I(\phi_1), \dots, I(\phi_{i-1})\}$ for any i .

Definition 9. (Consistency₃ of sequences of imperatives) The sequence $\langle I(\phi_1), \dots, I(\phi_n) \rangle$ is consistent₃ if and only if (1) The force structure of the set $\{I(\phi_1), \dots, I(\phi_n)\}$ has a consistent path; (2) The force structure of ϕ_i is compatible₃ with the force structure of the set $\{I(\phi_1), \dots, I(\phi_{i-1})\}$ for any i .

Definition 10. (Consistency₄ of sequences of imperatives) The sequence $\langle I(\phi_1), \dots, I(\phi_n) \rangle$ is consistent₄ if and only if (1) The force structure of the set $\{I(\phi_1), \dots, I(\phi_n)\}$ has a consistent path; (2) The force structure of ϕ_i is compatible₄ with the force structure of the set $\{I(\phi_1), \dots, I(\phi_{i-1})\}$ for any i .

Given the fact that $compatibility_2 < compatibility_3 < compatibility_4$, we have that $consistency_2 < consistency_3 < consistency_4$. Furthermore, it is easy to prove that $consistency_2$ is strictly stronger than $consistency_1$. Here are some typical examples, by which the four consistencies can be distinguished from each other:

- (16) a. Don't open the door! Open the door or the window!
 b. Open the door or the window! Don't open them both!
 c. Open the door! Open the door or the window!
 d. Open the door or turn on the TV! Open the door or the window!

By verifying these examples, we have the following results: (1) The sequence (16a) is only consistent₁; (2) (16b) is consistent₁ and consistent₂; (3) (16c) is consistent₁, consistent₂ and consistent₃; (4) (16d) is consistent under all these definitions. In addition, the antecedent and consequent of Ross's paradox is not consistent₄, but consistent₃ and consistent₂.

Previously we have independently defined the consistency of single imperatives. Here we generally define the consistency of sequences, which may contain only one imperative. Actually there is no conflict. Let $I(\phi)$ be any imperative. We can show that for any $i \leq 4$, the single sequence $\langle I(\phi) \rangle$ is consistent _{i} if and only if the force structure of $I(\phi)$ has a consistent path, which means that all these consistencies collapse to one when restricted to single imperatives.

In order to define the entailment for $update_1$ in section 5, we need to define the $consistency_1$ in terms of compatibility.

Definition 11. (Compatibility₁ of force structures) The force structures K_1 and K_2 are compatible₁ if and only if (1) If K_1 has a consistent path, then $K_1 \cup K_2$ has a consistent path; (2) If K_2 has a consistent path, then $K_1 \cup K_2$ has a consistent path.

Definition 12. (Consistency_{1'} of sequences of imperatives) The sequence $\langle I(\phi_1), \dots, I(\phi_n) \rangle$ is consistent_{1'} if and only if (1) The force structure of the set $\{I(\phi_1), \dots, I(\phi_n)\}$ has a consistent path; (2) The force structure of ϕ_i is compatible₁ with the force structure of the set $\{I(\phi_1), \dots, I(\phi_{i-1})\}$ for any i .

It is easy to see that $consistency_{1'}$ is equivalent to $consistency_1$, and $compatibility_1$ is weaker than $compatibility_2$. Note that according to this compatibility, inconsistent force structures are only compatible with inconsistent force structures and \emptyset , and \emptyset is compatible with any force structure.

In this section, we defined the notion of consistency of sequences of imperatives along two different directions. In the first direction we did not consider the restrictions between imperatives, and gave one definition. In the second direction, we defined three alternative notions of compatibility to characterize the restrictions, based on which three notions of consistency are defined. In the next section we will present semantics for imperatives based on these discussions.

4 Semantics

We propose that meaning of imperatives is an update function on force structures: Uttering imperatives will change force structures in some way. Let K be any force structure. Let $[\cdot]_1$, $[\cdot]_2$, $[\cdot]_3$ and $[\cdot]_4$ denote four update functions, which are defined as follows.

Definition 13. (Update₁ of force structures)

$$K[I(\phi)]_1 = \begin{cases} K \cup T^+(\emptyset, \phi) & K \text{ and } T^+(\emptyset, \phi) \text{ are consistent and compatible}_1 \\ \{\emptyset\} & \text{otherwise} \end{cases}$$

Definition 14. (Update₂ of force structures)

$$K[I(\phi)]_2 = \begin{cases} K \cup T^+(\emptyset, \phi) & K \text{ and } T^+(\emptyset, \phi) \text{ are consistent and compatible}_2 \\ \{\emptyset\} & \text{otherwise} \end{cases}$$

Definition 15. (Update₃ of force structures)

$$K[I(\phi)]_3 = \begin{cases} K \cup T^+(\emptyset, \phi) & K \text{ and } T^+(\emptyset, \phi) \text{ are consistent and compatible}_3 \\ \{\emptyset\} & \text{otherwise} \end{cases}$$

Definition 16. (Update₄ of force structures)

$$K[I(\phi)]_4 = \begin{cases} K \cup T^+(\emptyset, \phi) & K \text{ and } T^+(\emptyset, \phi) \text{ are consistent and compatible}_4 \\ \{\emptyset\} & \text{otherwise} \end{cases}$$

Recall that each force structure describes a state of imperative forces born by the agent, and for any $I(\phi)$, $T^+(\emptyset, \phi)$ is the corresponding force structure. When K and $T^+(\emptyset, \phi)$ are not absurd, updating K with $I(\phi)$ will put together K and $T^+(\emptyset, \phi)$, unless they are not compatible. If K or $T^+(\emptyset, \phi)$ are absurd, or they are not compatible, the update result is $\{\emptyset\}$, the absurd force structure.

The only difference among the four updates is that they require different conditions for two consistent force structures to be compatible, which can be expressed by the following inference: $K[I(\phi)]_1 = \{\emptyset\} \Rightarrow K[I(\phi)]_2 = \{\emptyset\} \Rightarrow K[I(\phi)]_3 = \{\emptyset\} \Rightarrow K[I(\phi)]_4 = \{\emptyset\}$.

In order to make clear how these semantics are running, and where they are different, we check the meaning of sequences in (16). We use the following formulas to represent these sequences:

$$(16') \text{ a. } I(\neg p_1); I(p_1 \vee p_2)$$

- b. $I(p_1 \vee p_2); I(\neg(p_1 \wedge p_2))$
- c. $I(p_1); I(p_1 \vee p_2)$
- d. $I(p_1 \vee p_3); I(p_1 \vee p_2)$

Updating \emptyset with these sequences will produce the following results:

- a. (1) $\emptyset[I(\neg p_1)]_1[I(p_1 \vee p_2)]_1 = \{\{\neg p_1\}\}[I(p_1 \vee p_2)]_1 = \{\{\neg p_1\}, \{p_1, p_2\}\}$
(2) $\emptyset[I(\neg p_1)]_2[I(p_1 \vee p_2)]_2 = \{\{\neg p_1\}\}[I(p_1 \vee p_2)]_2 = \{\emptyset\}$
(3) $\emptyset[I(\neg p_1)]_3[I(p_1 \vee p_2)]_3 = \{\{\neg p_1\}\}[I(p_1 \vee p_2)]_3 = \{\emptyset\}$
(4) $\emptyset[I(\neg p_1)]_4[I(p_1 \vee p_2)]_4 = \{\{\neg p_1\}\}[I(p_1 \vee p_2)]_4 = \{\emptyset\}$
- b. (1) $\emptyset[I(p_1 \vee p_2)]_1[I(\neg(p_1 \wedge p_2))]_1 = \{\{p_1, p_2\}\}[I(\neg(p_1 \wedge p_2))]_1 = \{\{p_1, p_2\}, \{\neg p_1, \neg p_2\}\}$
(2) $\emptyset[I(p_1 \vee p_2)]_2[I(\neg(p_1 \wedge p_2))]_2 = \{\{p_1, p_2\}\}[I(\neg(p_1 \wedge p_2))]_2 = \{\{p_1, p_2\}, \{\neg p_1, \neg p_2\}\}$
(3) $\emptyset[I(p_1 \vee p_2)]_3[I(\neg(p_1 \wedge p_2))]_3 = \{\{p_1, p_2\}\}[I(\neg(p_1 \wedge p_2))]_3 = \{\emptyset\}$
(4) $\emptyset[I(p_1 \vee p_2)]_4[I(\neg(p_1 \wedge p_2))]_4 = \{\{p_1, p_2\}\}[I(\neg(p_1 \wedge p_2))]_4 = \{\emptyset\}$
- c. (1) $\emptyset[I(p_1)]_1[I(p_1 \vee p_2)]_1 = \{\{p_1\}\}[I(p_1 \vee p_2)]_1 = \{\{p_1\}, \{p_1, p_2\}\}$
(2) $\emptyset[I(p_1)]_2[I(p_1 \vee p_2)]_2 = \{\{p_1\}\}[I(p_1 \vee p_2)]_2 = \{\{p_1\}, \{p_1, p_2\}\}$
(3) $\emptyset[I(p_1)]_3[I(p_1 \vee p_2)]_3 = \{\{p_1\}\}[I(p_1 \vee p_2)]_3 = \{\{p_1\}, \{p_1, p_2\}\}$
(4) $\emptyset[I(p_1)]_4[I(p_1 \vee p_2)]_4 = \{\{p_1\}\}[I(p_1 \vee p_2)]_4 = \{\emptyset\}$
- d. (1) $\emptyset[I(p_1 \vee p_3)]_1[I(p_1 \vee p_2)]_1 = \{\{p_1, p_3\}\}[I(p_1 \vee p_2)]_1 = \{\{p_1, p_3\}, \{p_1, p_2\}\}$
(2) $\emptyset[I(p_1 \vee p_3)]_2[I(p_1 \vee p_2)]_2 = \{\{p_1, p_3\}\}[I(p_1 \vee p_2)]_2 = \{\{p_1, p_3\}, \{p_1, p_2\}\}$
(3) $\emptyset[I(p_1 \vee p_3)]_3[I(p_1 \vee p_2)]_3 = \{\{p_1, p_3\}\}[I(p_1 \vee p_2)]_3 = \{\{p_1, p_3\}, \{p_1, p_2\}\}$
(4) $\emptyset[I(p_1 \vee p_3)]_4[I(p_1 \vee p_2)]_4 = \{\{p_1, p_3\}\}[I(p_1 \vee p_2)]_4 = \{\{p_1, p_3\}, \{p_1, p_2\}\}$

It should be pointed out that all these semantics except the first one don't have the property of commutativity. That is to say, for any i ($i = 2, 3, 4$), there are some K , $I(\phi)$ and $I(\psi)$ such that $K[I(\phi)]_i[I(\psi)]_i \neq K[I(\psi)]_i[I(\phi)]_i$. This implies that $update_2$, $update_3$ and $update_4$ are genuinely dynamic, while $update_1$ essentially is static. Next we give some examples to show that the commutativity does not hold for $update_2$, $update_3$ and $update_4$.

Let $K_1 = \{\{p_1, p_2\}\}$, $I(\phi_1) = I(\neg p_1 \vee \neg p_2)$ and $I(\psi_1) = I(p_1 \vee p_2 \vee p_3)$. It can be verified that $K_1[I(\phi_1)]_2[I(\psi_1)]_2 \neq \{\emptyset\}$ while $K_1[I(\psi_1)]_2[I(\phi_1)]_2 = \{\emptyset\}$, which means that $K_1[I(\phi_1)]_2[I(\psi_1)]_2 \neq K_1[I(\psi_1)]_2[I(\phi_1)]_2$.

Let $K_2 = \{\{p_1, p_2\}\}$, $I(\phi_2) = I(\neg p_1 \vee \neg p_4)$ and $I(\psi_2) = I(p_3 \vee p_4)$. We can verify that $K_2[I(\phi_2)]_3[I(\psi_2)]_3 \neq \{\emptyset\}$ while $K_2[I(\psi_2)]_3[I(\phi_2)]_3 = \{\emptyset\}$. It can also be verified that $K_2[I(\phi_2)]_4[I(\psi_2)]_4 \neq \{\emptyset\}$ while $K_2[I(\psi_2)]_4[I(\phi_2)]_4 = \{\emptyset\}$.

For any sequence $\langle I(\phi_1), \dots, I(\phi_n) \rangle$ and any $i \leq 4$, we have the following result: $\langle I(\phi_1), \dots, I(\phi_n) \rangle$ is consistent _{i} if and only if $\emptyset[I(\phi_1)]_i \dots [I(\phi_n)]_i \neq \{\emptyset\}$. This

means that updating the minimal force structure \emptyset with an inconsistent sequence results in an absurd force structure, which we think is reasonable.

The update function $[\cdot]$ defined in Veltman [13] is equivalent to $[\cdot]_2$ defined here in the following sense: For any plan Π and force structure K , if Π is the set of all consistent paths of K , then for any $I(\phi)$, $\Pi[\phi]$ is the set of all consistent paths of $K[I(\phi)]_2$.

5 Entailment

Let $\langle I(\phi_1), \dots, I(\phi_n) \rangle$ be any sequence, and $I(\psi)$ be any imperative. In this section we define $\langle I(\phi_1), \dots, I(\phi_n) \rangle \models I(\psi)$.

For any force structures K_1 and K_2 , we use $K_1 \approx_2 K_2$, $K_1 \approx_3 K_2$ and $K_1 \approx_4 K_2$ to respectively express that K_1 and K_2 have the same sets of consistent paths, routes and tracks. Relative to each semantics except the first one presented in the last section, we define an acceptance relation \vdash . The acceptance relation for the first semantics will be given later. Let K be any force structure, and $I(\phi)$ be any imperative.

Definition 17. (Acceptance₂) $K \vdash_2 I(\phi) \Leftrightarrow K \approx_2 K[I(\phi)]_2$.

Definition 18. (Acceptance₃) $K \vdash_3 I(\phi) \Leftrightarrow K \approx_3 K[I(\phi)]_3$.

Definition 19. (Acceptance₄) $K \vdash_4 I(\phi) \Leftrightarrow K \approx_4 K[I(\phi)]_4$.

That a force structure accepts an imperative means that updating this force structure with this imperative does not change the set of its consistent paths (routes or tracks). In particular, inconsistent force structures accept any imperatives, and the minimal force structure \emptyset only accepts inconsistent imperatives.

The difference among the three acceptance relations is a bit complicated. Simply speaking, it is not the case that for any K and $I(\phi)$, $K \vdash_2 I(\phi) \Rightarrow K \vdash_3 I(\phi) \Rightarrow K \vdash_4 I(\phi)$ or $K \vdash_4 I(\phi) \Rightarrow K \vdash_3 I(\phi) \Rightarrow K \vdash_2 I(\phi)$. But in some special cases, the strength of these acceptance relations is comparable. We look at the following proposition, whose proof is put to the appendix section.

Proposition 5.1. $K_1 \approx_2 K_2 \Rightarrow K_1 \approx_3 K_2 \Rightarrow K_1 \approx_4 K_2$.

Note it is not necessary true that $K_1 \approx_4 K_2 \Rightarrow K_1 \approx_3 K_2$. We give a counter-example for this point. Let $K_1 = \{\{p_1\}, \{p_1, p_2\}\}$, and $K_2 = \{\{p_1\}, \{p_1, \neg p_2\}\}$. It can be verified that K_1 and K_2 have the same set of consistent tracks but different sets of consistent routes. It is also not true that $K_1 \approx_3 K_2 \Rightarrow K_1 \approx_2 K_2$. Here is a counter-example: $K_1 = \{\{p_1\}, \{p_1, p_2\}, \{p_2, p_3\}\}$, and $K_2 = \{\{p_1\}, \{p_2, p_3\}\}$. By this proposition, we can see that if for any $2 \leq i \leq 4$, K and $\emptyset[I(\phi)]_i$ are compatible _{i} , then $K \vdash_2 I(\phi) \Rightarrow K \vdash_3 I(\phi) \Rightarrow K \vdash_4 I(\phi)$. This relation is strict, because based on the above examples, we can construct counter-examples for $K \vdash_4 I(\phi) \Rightarrow K \vdash_3 I(\phi) \Rightarrow K \vdash_2 I(\phi)$. For the first case, we let $K = \{\{p_1\}, \{p_1, p_2\}\}$ and $I(\phi) = I(p_1 \wedge (p_1 \vee \neg p_2))$. We have $K \vdash_4 I(\phi)$, but do not have $K \vdash_3 I(\phi)$. For the second case, we can let $K = \{\{p_1\}, \{p_2, p_3\}\}$ and $I(\phi) = I(p_1 \wedge (p_2 \vee p_3) \wedge (p_1 \vee p_2))$. We also can see that if K is inconsistent, $K \vdash_2 I(\phi) \Leftrightarrow K \vdash_3 I(\phi) \Leftrightarrow K \vdash_4 I(\phi)$, simply because for any acceptance relation, an inconsistent force structure accepts everything.

By using acceptance relation, we can define the notion of validity for imperatives. In what follows, we give three definitions of entailment, which are respectively relative to $update_2$, $update_3$ and $update_4$.

Definition 20. (Entailment₂)

$$\langle I(\phi_1), \dots, I(\phi_n) \rangle \models_2 I(\psi) \Leftrightarrow \emptyset [I(\phi_1)]_2 \dots [I(\phi_n)]_2 \vdash_2 I(\psi).$$

Definition 21. (Entailment₃)

$$\langle I(\phi_1), \dots, I(\phi_n) \rangle \models_3 I(\psi) \Leftrightarrow \emptyset [I(\phi_1)]_3 \dots [I(\phi_n)]_3 \vdash_3 I(\psi).$$

Definition 22. (Entailment₄)

$$\langle I(\phi_1), \dots, I(\phi_n) \rangle \models_4 I(\psi) \Leftrightarrow \emptyset [I(\phi_1)]_4 \dots [I(\phi_n)]_4 \vdash_4 I(\psi).$$

About the relation between the three notions of validity, it can be easily shown that neither $\models_2 \leq \models_3 \leq \models_4$ nor $\models_4 \leq \models_3 \leq \models_2$ is the case. However, when the sequence $\langle I(\phi_1), \dots, I(\phi_n) \rangle$ is consistent_{*i*} and compatible_{*i*} with $I(\psi)$ for any $2 \leq i \leq 4$, we have $\models_2 \leq \models_3 \leq \models_4$. Of course, when $\langle I(\phi_1), \dots, I(\phi_n) \rangle$ is inconsistent_{*i*} for any *i*, we have $\models_2 = \models_3 = \models_4$, since inconsistent force structures accept any imperative.

We defined validity in the above by using the minimal force structure \emptyset . In fact these notions of validity can not be equivalently defined by using arbitrary force structure K , because the following result does not hold: For any $2 \leq i \leq 4$, $\emptyset [I(\phi_1)]_i \dots [I(\phi_n)]_i \vdash_i I(\psi)$ if and only if for any force structure K , $K [I(\phi_1)]_i \dots [I(\phi_n)]_i \vdash_i I(\psi)$. Here we give an universal counter-example for all these three cases: $K = \{\{\neg p, \neg r\}\}$, $I(\phi_1) = I(p \vee q)$, $I(\phi_2) = I(r \vee s)$, and $I(\psi) = I((p \vee q) \wedge (r \vee s))$. It can be verified that for any $2 \leq i \leq 4$, $\emptyset [I(\phi_1)]_i [I(\phi_2)]_i \vdash_i I(\psi)$, but $K [I(\phi_1)]_i [I(\phi_2)]_i \not\vdash_i I(\psi)$, simply because $K [I(\phi_1)]_i [I(\phi_2)]_i [I(\psi)]_i = \{\emptyset\}$, while $K [I(\phi_1)]_i [I(\phi_2)]_i \neq \{\emptyset\}$.

Similar to the classical entailment in the update framework, all the notions of entailment defined above are based on some invariance of “information states”. The main difference is that, here it is the set of consistent paths (routes or tracks) of a force structure that is invariant, not the force structure itself. The reason that we treat force structures, not the sets of paths (routes or tracks), as information states is that we can contrast the alternative semantics easily in this way.

Our way of defining entailment can be justified from another perspective. Let K_1 and K_2 be any force structures. Look at the relations \sqsupseteq_2 , \sqsupseteq_3 and \sqsupseteq_4 , which are defined as follows:

- (a) $K_1 \sqsupseteq_2 K_2$ if and only if (1) For any consistent path P_1 of K_1 , there is a consistent path P_2 of K_2 such that $P_2 \subseteq P_1$; (2) For any consistent path P_2 of K_2 , there is a consistent path P_1 of K_1 such that $P_2 \subseteq P_1$.
- (b) $K_1 \sqsupseteq_3 K_2$ if and only if (1) For any consistent route R_1 of K_1 , there is a consistent route R_2 of K_2 such that $R_2 \subseteq R_1$; (2) For any consistent route R_2 of K_2 , there is a consistent route R_1 of K_1 such that $R_2 \subseteq R_1$.
- (c) $K_1 \sqsupseteq_4 K_2$ if and only if (1) For any consistent track T_1 of K_1 , there is a consistent track T_2 of K_2 such that $T_2 \subseteq T_1$; (2) For any consistent track T_2 of K_2 , there is a consistent track T_1 of K_1 such that $T_2 \subseteq T_1$.

Roughly speaking, \sqsupseteq_i ($i = 2, 3, 4$) behaves like the subset relation. For instance, suppose $K_1 \sqsupseteq_2 K_2$. In this case all the consistent paths of K_1 will survive in $K_1 \cup K_2$. That is to say that any consistent path of K_1 is also a consistent path of $K_1 \cup K_2$. All the consistent paths of K_2 will also survive in $K_1 \cup K_2$, although in some indirect way: Each of them appears in some consistent path of $K_1 \cup K_2$. Similar situations happen to \sqsupseteq_3 and \sqsupseteq_4 . As mentioned previously, paths, routes and tracks can be treated as three types of free choices to perform force structures. Here $K_1 \sqsupseteq_i K_2$ means three different

things: (1) Every free choice of performing K_2 is a part of performing K_1 ; (2) K_2 does not block any free choice of K_1 in the sense that there is no consistent path (route or track) of K_1 , which is not a consistent path (route or track) of $K_1 \cup K_2$; (3) K_2 does not generate new free choices relative to K_1 in the sense that there is no consistent path (route or track) of K_2 , which does not occur in some consistent path (route or track) of $K_1 \cup K_2$.

Let K be any consistent force structure. The following propositions hold, whose proofs are put to the appendix:

Proposition 5.2. $K \vdash_2 I(\phi) \Rightarrow K \sqsupseteq_2 \emptyset[I(\phi)]_2$.

Proposition 5.3. $K \vdash_3 I(\phi) \Leftrightarrow K \sqsupseteq_3 \emptyset[I(\phi)]_3$.

Proposition 5.4. $K \vdash_4 I(\phi) \Leftrightarrow K \sqsupseteq_4 \emptyset[I(\phi)]_4$.

Based on these facts, we consider these acceptance relations reasonable. Consequently the entailments have reasonable intuitions. Note that $K \sqsupseteq_2 \emptyset[I(\phi)]_2 \Rightarrow K \vdash_2 I(\phi)$ may not hold. A counter-example is this: $K = \{\{p_1\}, \{p_2, p_3\}\}$ and $I(\phi) = I(p_1 \vee p_2)$.

Next we give some examples to show how these notions of entailment work and how they differ from each other.

- (19) a. Invite Mary or invite Mary and Jack!
 b. Invite Mary or invite Mary but not Jack!
- (20) a. Invite Mary and invite Jack or John!
 b. Invite Mary and invite Mary or Jack and invite Jack or John!

We see that for any i ($i = 2, 3, 4$), all the imperatives in (19) and (20) are consistent _{i} . We also see that for any i , (19a) and (19b) are compatible _{i} , and (20a) and (20b) are compatible _{i} . We can verify that (19a) \models_4 (19b). However, the literal “not inviting Jack” does not occur in the force structure of (19a), hence (19a) $\not\models_3$ (19b) and (19a) $\not\models_2$ (19b). In this way “ \models_4 ” is distinguished from “ \models_3 ” and “ \models_2 ”. We can also verify that (20a) \models_3 (20b) but (20a) $\not\models_2$ (20b), because putting together (20a) and (20b) will produce the consistent path $\{\text{inviting Mary, inviting Jack, inviting John}\}$, which is not even a path of (20a).

A typical example for *entailment*₂ is (21):

- (21) a. Open the door and the window!
 b. Open the door or the window!

Actually (21a) \models_2 (21b).

The fact that (19a) \models_4 (19b) should be payed more attention. “Not inviting Jack” is a literal occurring in the force structure of (19b), but not a literal occurring in the force structure of (19a). However, this inference is still valid. This is a characteristic which neither \models_2 nor \models_3 has.

We now turn to define validity for *update*₁. Previously we defined *compatibility*₁ by consistent paths in this way: For any K_1 and K_2 , they are compatible if and only if (1) If K_1 has a consistent path, then $K_1 \cup K_2$ has a consistent path; (2) If K_2 has a consistent path, then $K_1 \cup K_2$ has a consistent path. In deed, replacing “path” by “route” or “track” in this definition will not change this compatibility. Now we give three alternative notions of entailment for *update*₁, all of which are coincident with *compatibility*₁.

Definition 23. (Acceptance₁₂) $K \vdash_{12} I(\phi) \Leftrightarrow K \approx_2 K[I(\phi)]_1$.

Definition 24. (Acceptance₁₃) $K \vdash_{13} I(\phi) \Leftrightarrow K \approx_3 K[I(\phi)]_1$.

Definition 25. (Acceptance₁₄) $K \vdash_{14} I(\phi) \Leftrightarrow K \approx_4 K[I(\phi)]_1$.

Definition 26. (Entailment₁₂)

$$\langle I(\phi_1), \dots, I(\phi_n) \rangle \models_{12} I(\psi) \Leftrightarrow \emptyset[\phi_1]_1 \dots [\phi_n]_1 \vdash_{12} I(\psi).$$

Definition 27. (Entailment₁₃)

$$\langle I(\phi_1), \dots, I(\phi_n) \rangle \models_{13} I(\psi) \Leftrightarrow \emptyset[\phi_1]_1 \dots [\phi_n]_1 \vdash_{13} I(\psi).$$

Definition 28. (Entailment₁₄)

$$\langle I(\phi_1), \dots, I(\phi_n) \rangle \models_{14} I(\psi) \Leftrightarrow \emptyset[\phi_1]_1 \dots [\phi_n]_1 \vdash_{14} I(\psi).$$

Since $compatibility_1 < compatibility_2 < compatibility_3 < compatibility_4$, we can see that for any i ($i = 2, 3, 4$), $\vdash_{1i} < \vdash_i$, which means that the acceptance relation \vdash_{1i} is weaker than the acceptance relation \vdash_i . However, we don't have $\models_{1i} < \models_i$. The reason for this lies in that $\emptyset[\phi_1]_i \dots [\phi_n]_i$ is "easier" to be the absurd force structure $\{\emptyset\}$ than $\emptyset[\phi_1]_1 \dots [\phi_n]_1$, and $\{\emptyset\}$ entails anything. Essentially all the difference between \models_{1i} and \models_i is caused by the difference between $compatibility_1$ and $compatibility_i$.

We restate Ross's paradox as (23):

(23) Slip the letter into the letter-box! \models Slip the letter into the letter-box or burn it!

It can be verified that this inference is invalid relative to any notion of validity we have defined. The basic reason for this is that the consequent contains new free choice relative to the antecedent. Furthermore, the antecedent and consequent are not even compatible according to $compatibility_4$.

Finally we focus on such a question: What is the relation between the logic for imperatives, which are determined by the notions of entailment defined previously, and classical propositional logic, if we only consider the propositional contents of imperatives.

Let $I(\phi)$ be any imperative, and K be its force structure. The force structure K can be transformed to a conjunctive normal form (CNF) in the following way: (1) For any choice scope of K , connect all literals in it by disjunction and we will get a simple disjunction; (2) Connect all these simple disjunctions and we will get a CNF. It can be proved that the resulting CNF is equivalent to ϕ in the classical logic. Actually for any $\phi \in \mathcal{P}$, the definition of T^+ and T^- describes a procedure to get one of its CNFs.

For any K , let $\Phi(K)$ be the corresponding CNF of K . Let " \models " be the classical entailment. In fact we have such a proposition, whose proof is in the appendix section:

Proposition 5.5. For any i ($i = 2, 3, 4$), and for any K_1 and K_2 , $K_1 \approx_i K_1 \cup K_2 \Rightarrow \Phi(K_1) \models \Phi(K_2)$.

By this proposition we can prove that for any i , $\models_{1i} \subseteq \models$. This means that the logics for imperatives revealed by the three notions of entailment are subsets of classical logic. But for the entailment notions defined for $update_2$, $update_3$ or $update_4$, this is not the case. Actually the following results hold: For any i , $\models_i \not\subseteq \models$ and $\models \not\subseteq \models_i$. For instance, Ross's paradox is valid in classical logic, but invalid according to these notions of entailment. On the other side, any inconsistent sequence of imperatives, whose force structure is consistent, entails any imperative. But this is not the case in the classical logic.

6 Conclusion

The aim of this paper is to propose solutions to consistency problem of imperatives and Ross's paradox. Imperatives exert imperative forces to the agent. Each force structure represents a state of imperative forces born by the agent. Under the function T^+ , each imperative corresponds to a force structure. Uttering an imperative will add some forces to the previous force structure, and get a new force structure. The meaning of imperatives lies in how they change force structures. This is our way to develop semantics for imperatives in the framework of update semantics.

As Veltman [13] says, it is not always clear where in the field of imperatives the borderline between semantics and pragmatics should be drawn. In this paper, this problem is embodied as how we treat compatibility between imperatives: Whether the compatibility between imperatives should be put into semantics has to be considered. We didn't make a choice at this point. What we did is to define consistency of sequences of imperatives along each of the two directions. When we count compatibility in semantics, in deed there are different alternative definitions of compatibility, each of which corresponds to one type of free choices. It is not clear which one is the best. We didn't make any choice among these candidates, and just sorted out them. Based on different notions of compatibility, we gave several different definitions of semantics for imperatives. Keeping coincident with semantics, we defined different notions of validity relative to specific semantics. Ross's paradox is invalid with respect to any of these notions of validity.

At the beginning of this paper, we mentioned that some boolean combinations containing imperatives are meaningful, such as pseudo-imperatives and conditional imperatives. Based on the work in this paper, it is promising to develop an uniform semantics to deal with these language phenomena. This will be our future work.

Appendix

In this section we give the proofs of some propositions mentioned previously. For simplifying the statements, we stipulate some general abbreviations. $K_1 \bowtie_i K_2$ means that K_1 and K_2 are *compatible_i*. $P \triangleright K$, $R \triangleright K$ and $T \triangleright K$ respectively express that P is a consistent path of K , R is a consistent route of K , and T is a consistent track of K . For any $I(\phi)$, we have $\emptyset[I(\phi)]_1 = \emptyset[I(\phi)]_2 = \emptyset[I(\phi)]_3 = \emptyset[I(\phi)]_4$. We use $[\phi]$ to represent $\emptyset[I(\phi)]_i$.

The following three lemmas will be used freely in this section. We only prove the first one, and the other two can be proved in a similar way.

Lemma 3.1. *Let K_1 and K_2 be any non-empty force structures. For any consistent path P_1 of K_1 , there is a consistent path P of $K_1 \cup K_2$ such that $P_1 \subseteq P$ if and only if there is a consistent path P_2 of K_2 such that $P_1 \cup P_2$ is consistent.*

Proof. (1) \Leftarrow

This direction is trivial.

(2) \Rightarrow

Let $P_1 \triangleright K_1$. Suppose $P \triangleright K_1 \cup K_2$ such that $P_1 \subseteq P$. Hence there are $P_2 \triangleright K_1$ and $P_3 \triangleright K_2$ such that $P = P_2 \cup P_3$. Then $P_1 \subseteq P_2 \cup P_3$. Then $P_1 \cup P_3 \subseteq P$. Then $P_1 \cup P_3$ is consistent. □

Lemma 3.2. *Let K_1 and K_2 be any non-empty force structures. For any consistent route R_1 of K_1 , there is a consistent route R of $K_1 \cup K_2$ such that $R_1 \subseteq R$ if and only if there is a consistent route R_2 of K_2 such that $R_1 \cup R_2$ is consistent.*

Lemma 3.3. *Let K_1 and K_2 be any non-empty force structures. For any consistent track T_1 of K_1 , there is a consistent track T of $K_1 \cup K_2$ such that $T_1 \subseteq T$ if and only if there is a consistent track T_2 of K_2 such that $T_1 \cup T_2$ is consistent.*

Proposition 5.1. $K_1 \approx_2 K_2 \Rightarrow K_1 \approx_3 K_2 \Rightarrow K_1 \approx_4 K_2$.

Proof. (1) $K_1 \approx_2 K_2 \Rightarrow K_1 \approx_3 K_2$

For any force structure, each consistent route of it is the union of some consistent paths of it. The same set of consistent paths generate the same set of consistent routes. Hence $K_1 \approx_2 K_2 \Rightarrow K_1 \approx_3 K_2$.

(2) $K_1 \approx_3 K_2 \Rightarrow K_1 \approx_4 K_2$

Suppose $K_1 \approx_3 K_2$.

Assume that neither K_1 nor K_2 contains any consistent route. Clearly $K_1 \approx_4 K_2$ in this case.

Now assume that both of them contain consistent routes.

Firstly we show that any propositional variable occurring in K_1 also occurs in K_2 , and vice versa. Let p be any variable. Suppose that p appears in K_1 . Then there is a choice scope $J \in K_1$ such that p or $\neg p$ is in J . We claim that p or $\neg p$ must appear in some consistent route R . Assume this is not the case. Let R_1 be any consistent route of K_1 . Then $R_1 \cup \{p\}$ or $R_1 \cup \{\neg p\}$ is a consistent route of K_1 , which results in a contradiction. Since $K_1 \approx_3 K_2$, we know that the variable p appears in K_2 . Similarly we can show that any variable occurring in K_2 also occurs in K_1 .

Suppose $T_1 \triangleright K_1$. Then $T_1 \cap (\bigcup K_1)$ is a consistent route of K_1 . Hence $T_1 \cap (\bigcup K_1)$ is a consistent route of K_2 . Since K_1 and K_2 share the same set of variables occurring in them, we know that $T_1 \triangleright K_2$. Similarly we can prove that any consistent track of K_2 is also a consistent track of K_1 . □

In order to prove **Proposition 5.2**, **Proposition 5.3** and **Proposition 5.4**, we need the following lemmas, whose proofs are simply omitted.

Lemma 5.1. *For any non-empty force structures K_1 and K_2 , $P \triangleright K_1 \cup K_2$ if and only if there are $P_1 \triangleright K_1$, and $P_2 \triangleright K_2 - K_1$ such that $P_1 \cup P_2$ is consistent and $P = P_1 \cup P_2$.*

Lemma 5.2. *For any non-empty force structures K_1 and K_2 , $R \triangleright K_1 \cup K_2$ if and only if there are $R_1 \triangleright K_1$, and $R_2 \triangleright K_2$ such that $R_1 \cup R_2$ is consistent and $R = R_1 \cup R_2$.*

Lemma 5.3. *For any non-empty force structures K_1 and K_2 , $T \triangleright K_1 \cup K_2$ if and only if there are $T_1 \triangleright K_1$, and $T_2 \triangleright K_2$ such that $T_1 \cup T_2$ is consistent and $T = T_1 \cup T_2$.*

Proposition 5.2. *For any consistent K , $K \vdash_2 I(\phi) \Rightarrow K \sqsupseteq_2 \emptyset[I(\phi)]_2$.*

Proof. Suppose $K \vdash_2 I(\phi)$. Then $K \bowtie_2 [\phi]$, and $K \approx_2 K[I(\phi)]_2$, that is, $K \approx_2 K \cup [\phi]$.

Let $P_1 \triangleright K$. $P_1 \triangleright K \cup [\phi]$, since $K \approx_2 K \cup [\phi]$. Hence there are $P_3 \triangleright K_1 - [\phi]$ and $P_2 \triangleright [\phi]$ such that $P_3 \cup P_2$ are consistent, and $P_1 = P_3 \cup P_2$. Clearly $P_2 \subseteq P_1$.

Let $P_2 \triangleright [\phi]$. Since $K \bowtie_2 [\phi]$, there is $P_3 \triangleright K$ such that $P_3 \cup P_2$ is consistent. Then there is $P_5 \triangleright K - [\phi]$ and $P_5 \subseteq P_3$. Then $P_5 \cup P_2$ is consistent. Then $P_5 \cup P_2 \triangleright K \cup [\phi]$. Because $K \approx_2 K \cup [\phi]$, we have $P_5 \cup P_2 \triangleright K$. Clearly $P_2 \subseteq P_5 \cup P_2$. \square

Proposition 5.3. For any consistent K , $K \vdash_3 I(\phi) \Leftrightarrow K \sqsupseteq_3 \emptyset[I(\phi)]_3$.

Proof. (1) \Rightarrow

The proof of this direction is similar to the proof of Proposition 5.2.

(2) \Leftarrow

Suppose $K \sqsupseteq_3 \emptyset[I(\phi)]_3$. It can be seen that $K \bowtie_3 [\phi]$. So it suffices to prove $K \approx_3 K[I(\phi)]_3$, that is, $K \approx_3 K \cup [\phi]$.

Let $R_1 \triangleright K$. Then there is $R_2 \triangleright [\phi]$ such that $R_1 \cup R_2 = R_1$. Clearly $R_1 \triangleright K \cup [\phi]$.

Now let $R \triangleright K \cup [\phi]$. Then there are $R_1 \triangleright K$ and $R_2 \triangleright [\phi]$ such that $R = R_1 \cup R_2$. It can be seen that any literal occurring in some consistent route of $[\phi]$ also occurs in some consistent route of K . Therefore $R_1 \cup R_2 \triangleright K$, that is, $R \triangleright K$. \square

Proposition 5.4. For any consistent K , $K \vdash_4 I(\phi) \Leftrightarrow K \sqsupseteq_4 \emptyset[I(\phi)]_4$.

Proof. (1) \Rightarrow

The proof of this direction is similar to the proof of Proposition 5.2.

(2) \Leftarrow

Suppose $K \sqsupseteq_4 \emptyset[I(\phi)]_4$. We can see that $K \bowtie_4 [\phi]$. Now we prove $K \approx_4 K \cup [\phi]$.

Let $T_1 \triangleright K$. Then there is $T_2 \triangleright [\phi]$ such that $T_1 \cup T_2 = T_1$. Therefore $T_1 \triangleright K \cup [\phi]$.

Now let $T \triangleright K \cup [\phi]$. Then there are $T_1 \triangleright K$ and $T_2 \triangleright [\phi]$ such that $T = T_1 \cup T_2$.

We can see that any variable occurring in $[\phi]$ also occurs in K . Hence $T_2 \subseteq T_1$. Hence $T = T_1 \cup T_2 \triangleright K$. \square

Proposition 5.5. For any i ($i = 2, 3, 4$), and for any K_1 and K_2 , $K_1 \approx_i K_1 \cup K_2 \Rightarrow \Phi(K_1) \models \Phi(K_2)$.

Proof. We only prove when $i = 2$, this is the case. Other proofs are similar to the following one.

Let \mathcal{A} be the set of literals of \mathcal{L} . $\Theta \subseteq \mathcal{A}$ is a valuation if and only if for any variable $p \in \mathcal{L}$, one and only one of p and $\neg p$ is in Θ . We can see that a valuation essentially is a classical model. As we know, $\Phi(K_1) \models \Phi(K_2)$ if and only if for any Θ , if Θ is a model of $\Phi(K_1)$, then Θ is also a model of $\Phi(K_2)$. For any Θ and K , Θ is a model of $\Phi(K)$ if and only if there is $P \triangleright K$ such that $P \subseteq \Theta$. Therefore $\Phi(K_1) \models \Phi(K_2)$ if and only if for any Θ , if there is $P_1 \triangleright K_1$ such that $P_1 \subseteq \Theta$, then there is $P_2 \triangleright K_2$ such that $P_2 \subseteq \Theta$.

Suppose $K_1 \approx_2 K_1 \cup K_2$. Let Θ be any valuation. Suppose $P_1 \triangleright K_1$ and $P_1 \subseteq \Theta$. According to the proof of Proposition 5.2, there is $P_2 \triangleright K_2$ such that $P_2 \subseteq P_1$. Clearly $P_2 \subseteq \Theta$. Hence $\Phi(K_1) \models \Phi(K_2)$. \square

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